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LETTER TO THE EDITOR

$Z(2)$ gauge model on fractal lattices

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Abstract. Hamiltonians with $Z(2)$ gauge symmetry are studied on a class of fractal lattices. The lattices considered here have Fourier and Hausdorf–Besicovitch dimensions strictly greater than 2, but they have a low degree of connectivity characterised by their connectivity index 2. It is proved that there is no phase transition at non-zero temperatures.

The study of critical behaviour of Hamiltonians on fractal lattices is interesting *per se*, and may help in the understanding of phase transitions in general. Earlier work in this field has been mostly for spin Hamiltonians with global symmetry (the n-vector model, or the k-state Potts model) (Nelson and Fisher 1975, Gefen *et al* 1980). In this Letter we study the behaviour of Hamiltonians with local $Z(2)$ symmetry on some fractal lattices.

The $Z(2)$ gauge model in two dimensions (say on a square lattice with simple plaquette interactions) is trivial and does not undergo any phase transitions. In order to have a non-trivial model, it is necessary to consider larger loop interactions (Gruber *et al* 1977), or to consider lattices which are somewhat better connected than the square lattice. The $Z(2)$ gauge model on a three-dimensional simple cubic lattice is equivalent to the Ising model on the same lattice, which is a well known unsolved problem. $Z(2)$ models on fractal lattices of dimensions between two and three are problems of intermediate complexity between these two extremes.

For the sake of definiteness, let us consider a fractal lattice L, defined as the direct product of the linear chain graph (lattice) L_1, and a modified rectangular lattice L_2. (For a definition of the direct product and of the modified rectangular lattice, see Dhar (1977).) This lattice may be obtained from a simple cubic lattice by deleting some links (bonds). The Hausdorf–Besicovitch dimension of this lattice is three, while its Fourier dimension is $\frac{5}{2}$. The lattice is invariant with respect to translations along L_1 (hereafter referred to as the time direction). The elementary loops of this lattice are of two kinds. On a constant 'time' hypersurface, there are the elementary loops of the modified rectangular lattice L_2, these have perimeters 4, 8, 16, ... The other kind of elementary loops are of length 4, and include two time-like links (see figure 1).

We assume an Ising spin S_B^Z taking values ± 1, at each of the links B of the lattice L. Let $B_1, B_2, \ldots, B_l$ be the links constituting an elementary loop p. We define the plaquette variable S_p^Z by

$$S_p^Z = S_{B_1}^Z S_{B_2}^Z \ldots S_{B_l}^Z. \tag{1}$$

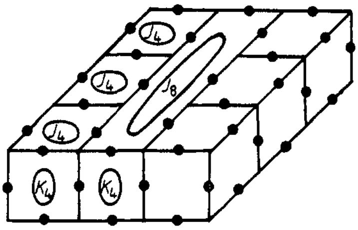

Figure 1. A portion of the lattice L. The spins (full circles) are situated on the links of the lattice. Some of the space-like loops of length 4 and 8 (coupling constants J_4 and J_8) and two of the non-space-like loops (coupling constants K_4) are shown.

The Hamiltonian of the $Z(2)$ model is defined to be

$$H = -\sum_p J_p S_p^Z - \sum_q K_q S_q^Z. \tag{2}$$

Here the summation over p extends over all the space-like elementary loops p, and the summation over q extends over the second kind (non space-like) of elementary loops. J_p and K_q are interaction constants.

The Hamiltonian is clearly unchanged if spins on all the links meeting at any particular point are reversed. This is the local symmetry of this model. The duality transformation for this lattice is identical to that for a simple cubic lattice (Gruber *et al* 1977, Savit 1980) with unequal plaquette interactions. Just as for the cubic lattice, the high-temperature series expansion for the partition function of the Hamiltonian H can be shown to be the same as the low-temperature expansion of an Ising Hamiltonian H^{D}, defined on a lattice L^{D}, the dual of L. L^{D} is quite easily seen to be the direct product of L_1 with $\mathrm{L}_2^{\mathrm{D}}$, the dual lattice of L_2. The sites of the dual lattice L^{D} correspond to elementary 'boxes' of the original lattice L. Two sites in L^{D} are nearest neighbours, if the corresponding boxes in L share a common plaquette. The Hamiltonian H^{D} is of the form

$$H^{\mathrm{D}} = \sum_{\langle ij \rangle} K_{ij} S_i S_j. \tag{3}$$

Here S_i is the Ising spin at the site i of the dual lattice L^{D}, and the summation extends over all nearest-neighbour sites i and j. Explicit expressions for K_{ij} in terms of J_p and K_q are easy to write down.

We now show that a system described by the Hamiltonian H^{D} does not undergo any phase transition as a function of temperature, and always exists in the ordered phase. We observe that a nearest-neighbour Ising model with spins on the vertices of the lattice L_2, and arbitrary but bounded nearest-neighbour interactions, is always disordered at any non-zero temperature. The duality transformation maps the high-temperature phase of this Hamiltonian (say H_2) defined on L_2, to the low-temperature phase of a nearest-neighbour Ising Hamiltonian H_2^{D} defined on $\mathrm{L}_2^{\mathrm{D}}$. This implies that H_2^{D} defined on $\mathrm{L}_2^{\mathrm{D}}$ is always in the low-temperature ordered phase, and the corresponding critical temperature is infinite. (All the bond strengths on this lattice $\mathrm{L}_2^{\mathrm{D}}$ are finite. The critical temperature is infinite due to the existence of vertices with arbitrarily large coordination numbers.) If from the lattice L^{D} we delete all the time-like links, it breaks up into mutually non-interacting layers $\mathrm{L}_2^{\mathrm{D}}$, the Hamiltonian of each layer being of the form H_2^{D}. As the critical temperature can only increase by introducing intra-layer ferromagnetic couplings (Griffiths' inequality), it is infinite for the Hamiltonian H^{D} also, and the corresponding system is always in the ordered phase. Again, by duality it follows that the original Hamiltonian H always describes the disordered phase.

Note that the proof holds for arbitrary non-negative bounded values of the interaction constants J_p and K_q. The boundedness constraint can be relaxed somewhat. For example, we may take K_q to be unbounded (or strictly infinite) for all plaquettes q not sharing any common links with a space-like elementary loop of perimeter 4, without affecting the argument.

It is quite easy to show that in the continuum–time limit, the transfer matrix for the Hamiltonian H can be written as the exponential of a quantum mechanical Hamiltonian H_{QM}

$$H_{\mathrm{QM}} = -\sum_p J_p S_p^Z - \sum_B h_B S_B^X \tag{4}$$

where H_{QM} is the Hamiltonian for a system of interacting quantum mechanical spins defined on the links of the lattice L_2. S_{B}^X is the spin–flip operator for the spin at link B. The h_{B} are interaction constants related to K_q. The non-existence of a phase transition for the Hamiltonian H implies that for the Hamiltonian H_{QM}, for arbitrarily small transverse fields h_{B}, the ground state is always disordered.

The proof is easily generalised to other fractal lattices L of the type $\mathrm{L} = \mathrm{L}_1 \times \mathrm{L}_2$, where L_1 is the linear chain graph, and L_2 is a planar graph of connectivity index 1. For example, L_2 may be the $(m, 1)$ recursive lattice (Dhar 1980), or the truncated tetrahedron lattice (Nelson and Fisher 1975). The definition of the connectivity index is given in Dhar (1980). For these lattices, the Hausdorf–Besicovitch and the Fourier dimensions are strictly greater than 2. However, the connectivity index for these lattices is 2. The non-existence of phase transitions in these gauge models would be inconsistent with the generally accepted value 2 for the lower critical dimension of models with discrete gauge symmetry, if we used the Hausdorf–Besicovitch or Fourier definitions of dimension. Clearly, for H Hamiltonians with discrete degrees of freedom, for local as well as for global symmetries, the lower critical dimension is best defined in terms of the connectivity index.

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